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SUBCRITICAL AND FATIGUE CRACK GROWTH IN ANTI-PLANE STRAIN STATE

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Abstract—Assuming elastic-plastic material behavior the slow growth of Mode III crack under both monotonic and pulsating loadings is considered. Rice's idea of universal R-curve is employed while the mathematical analysis is based on the one-dimensional plasticity model suggested by Kostrov and Nikitin. Motion of a quasi-static Mode III crack is studied and the stable/unstable transition points are found through application of the final stretch failure condition proposed in 1972 by Wnuk. A logarithmic formula for fatigue crack extension rate is derived. Results are compared to other well-known solutions.

NOTATION

τ_∞ applied stress at infinity
τ_0 yield stress in shear
$Q = \tau_\infty/\tau_0$ loading parameter
R plastic zone size
l half crack length
μ Kirchhoff's modulus
Δ size of Neuber's particle
R_i, R_∞ initial and asymptotic values of plastic zone size, respectively
α ductility parameter
$A_1, A_2, A_3, A_4, A_5, \beta(=1-2\alpha)$ constants
γ $\sqrt{(\alpha^2+4\alpha)}$
A_1 $\dfrac{5-4\alpha}{2(1-2\alpha)^2}$
A_2 $\dfrac{3}{2\cdot\beta^2}$
A_3 $-\dfrac{1}{2}$
A_4 $\dfrac{(\alpha^2-\alpha+1)\cdot\gamma+\alpha\cdot(\alpha^2+\alpha-3)}{\beta^2\cdot\gamma}$
A_5 $\dfrac{(\alpha^2-\alpha+1)\cdot\gamma+\alpha\cdot(3-\alpha-\alpha^2)}{\beta^2\cdot\gamma}$
v_o $\sqrt{\left(\dfrac{\rho_0-1}{\rho_0}\right)}$
v_1 $\dfrac{\alpha-\sqrt{(\alpha^2+4\alpha)}}{2}$
v_2 $\dfrac{\alpha+\sqrt{(\alpha^2+4\alpha)}}{2}$
K_{max}, K_{min} maximum and minimum stress intensity factors during a cycle, respectively
u_0, R_0 final stretch and related to it opening constant, $R_0 = (\mu/\tau_0)u_0$
K_c fracture toughness
N number of cycles
ρ_{max}, ρ_{min} maximum and minimum non-dimensional plastic zone sizes during a loading cycle, respectively ($\rho = R/\Delta$).

1. INTRODUCTION

The primary objective of the present paper is to find the stable/unstable transition in the failure mode for a material containing crack in anti-plane strain under monotonic and cyclic loadings. In the first case, the crack eventually becomes unstable leading to the loss of structural integrity

✠Dr. Marian Piszczek died tragically at the age of 30 on 23 November 1978 in Illinois.

of the whole component, while in the second problem it is required that the fatigue crack extension rate or the life of an element are predicted from the theoretical considerations.

First we shall derive an equation of motion of a subcritical growth of crack which is assumed to be quasi-static up to the point of terminal instability, i.e. catastrophic fracture. In order to use Rice's concept of a resistance-curve[1] one needs to know the fields of plastic stress and strain at the crack tip. It will be assumed, that the extent of yielding is small compared to the crack length, i.e we shall contain attention to the small scale yielding case. The plastic flow condition that we shall require to be satisfied is the Huber–Mises–Hencky maximum shear strain energy criterion. A modified Dugdale model equipped with the final stretch failure criterion will be applied in order to make the problem mathematically tractable.

One-dimensional model of plastic zone in longitudinal shear, see Kostrov–Nikitin[2], gives for the small scale yielding range and for a stationary crack

$$\frac{R}{l} = 2Q^2. \tag{1}$$

This may be compared with other known results, namely Hult–McClintocks:†

$$\frac{R}{l} = Q^2 \tag{1a}$$

and Dugdale $- B - C - S$:

$$\frac{R}{l} = 1.23Q^2. \tag{1b}$$

Such comparison seems to be encouraging. The R/l ratio is proportional to the square of the loading parameter in all of the above solutions. It is clear, that the extent of the plastic zone directly ahead of the crack does not appear to be very sensitive to the particular shape of the yielded zone associated with a Mode III crack.

Let us now briefly discuss the plastic yielding around a stationary crack according to the model of Kostrov and Nikitin[2]. We recall first that in applying the Dugdale model of plastic flow at the vicinity of the crack in anti-plane strain state, some singularity in τ_{xz} appears which influences the COD value and the plastic zone length. (Such singularity does not exist for the plane stress state.) Using the Dugdale model for anti-plane strain situation one obtains in the region near the crack tip

$$\tau_{xz} > \tau_0. \tag{2}$$

This fact was pointed out by Heald[3] and Field[4]. Solutions given by Heald[3] and by Field[4] indicate the singular behaviour of τ_{xz} at the crack tip. On the other hand, the one-dimensional model of Kostrov and Nikitin[2] tends to bear this singularity out and to eliminate it. These authors require the Huber–Mises–Hencky plastic condition to be satisfied

†The formula $R = lQ^2$ resulted from expanding the exact expression of Hult and McClintock for the extent of the plastic zone directly ahead of the crack tip

$$R = l\left\{\frac{2}{\pi}\frac{1+Q^2}{1-Q^2}E\left(\frac{2Q}{1+Q^2}\right) - 1\right\} \tag{1*}$$

at the point $Q = 0$. Symbol E denotes a complete elliptic integral of the second kind, while $(2Q/1 + Q^2)$ is its argument. The expression (1*) was found by Bilby, Cottrell and Swinden[15] and compared against their result

$$R = l\{\sec(\pi Q/2) - 1\}, \tag{1**}$$

which happens to be identical with Dugdale formula obtained for a Mode I crack. The differences between (1*) and (1**) do not exceed 5%.

within all of the yielded zone associated with a Mode III crack, namely

$$\sqrt{\tau_{xz}^2 + \tau_{yz}^2} = \tau_0. \tag{3}$$

It should be pointed† out, however, that the Kostrov–Nikitin's assumption of one-dimensional plastic zone subjected to the Mises yield criterion violates the associated plastic flow rule unlike the Heald's and Field's solutions which employ the one-dimensional plasticity model combined with the Tresca yield criterion and the classical solution of Hult and McClintock[13] derived from field plasticity coupled with the von Mises yield condition. Although some exceptions to the associated plastic flow rule may exist and they have been observed experimentally, see [16], it is generally believed that such associated flow rule is valid for any homogeneous elastic–plastic material.

Under the assumption (3) the mathematical analysis results in the displacement distribution along the plastic zone (see Fig. 1):

$$u = \pm \frac{l(\tau_0^2 + \tau_\infty^2)}{2\mu\tau_\infty}\left(\frac{1}{2}\log\frac{a - \sqrt{(a^2 - x^2)}}{a + \sqrt{(a^2 - x^2)}} + \frac{\sqrt{(a^2 - x^2)}}{a}\right) \tag{4}$$

for $y = \pm 0, \quad l \leq x \leq a.$

Dimensionless function (4) is shown in Fig. 2.

2. DIFFERENTIAL EQUATION GOVERNING THE STABLE CRACK GROWTH. FINAL STRETCH CRITERION

The definition of "stable growth", even though very often used, isn't adequate because any growth leading to catastrophe is unstable in the local sense while the terminal (critical) point corresponds to the loss of stability in a global sense. Irwin and McClintock[5] showed, that for ductile materials subcritical crack growth proceeds while the applied load increases. Fracture extension of this type can be decribed in two ways:

(a) through a variable stress intensity factor K, which depends on the current crack length;

(b) through a resistance curve, either an R-curve of a J_R-curve. Both approaches reduce to

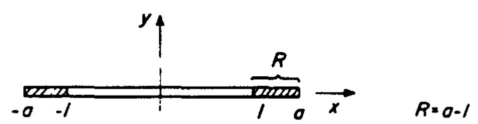


Fig. 1. One-dimensional model of the plastic deformation zones associated with a Mode III crack. The stresses within the yield zone are required to satisfy the Miese-Huber-Hencky criterion $\tau_{xz}^2 + \tau_x^2 = \tau_0^2$.

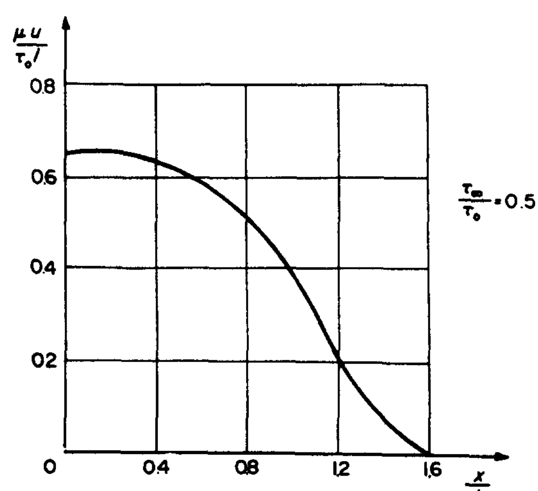


Fig. 2. Crack "profile" according to the Kostrov-Nikitin model (the z-component of the displacement field is shown).

†Authors are indebted to the reviewer who brought this point out.

certain nonlinear differential equations, for $R(l)$ or $K(l)$ functions. Rice was the first to propose the variable length of plastic zone as a factor controlling subcritical crack growth. Idea of Rice was then shown equivalent to the approach of Irwin and McClintock, see [1].

Fracture criterion proposed by Wnuk[17] is somewhat analogous to the Cherepanov's[7] hypothesis of a constant plastic energy dissipation but it is only the fraction of the total energy dissipated within the plastic zone, named the "essential work of fracture" (lost in the final act of fracture occurring over the Neuber's particle) which is supposed to control the quasi-static crack growth. Wnuk required that the prior-to-fracture work done at a fixed material point P, while the process zone of micro-structural dimension Δ passes through it, is a material property

$$\int_{t-\delta t}^{t} S(x_p, \tau) \cdot \dot{u}(x_p, \tau)\,\mathrm{d}\tau = \text{constant}. \tag{5}$$

Since the length Δ is on the order of a characteristic micro-structural size, sometimes referred to as "Neuber's particle" (approaching zero for a perfectly brittle continuum), one may apply the approximation of Glennie and Willis[8] and divide the unsteady motion of crack into a number of constant speed segments. With an assumption of a constant restraining stress equal yield stress Wnuk's criterion for failure reads

$$\int_{t-\delta t}^{t} \dot{u}(x_p, \tau)\,\mathrm{d}\tau = u_0 \tag{6}$$

in where the symbol u_0 denotes the opening constant or the "final stretch". Final stretch and COD criteria are equivalent only in the limit, when $\delta R \to 0$, that is, when R approaches its steady state value R_∞.

Developing Kostrov–Nikitin's solution (4) into Taylor expansion we arive at

$$[u_p{}^{+}] \underset{x\to 0}{=} \frac{\sqrt{2}(\tau_0{}^2 + \tau_\infty{}^2)}{2\mu\tau_\infty} \frac{(R-x)^{3/2}}{\sqrt{l}} + \cdots \tag{7}$$

Here x denotes the distance measured from the crack tip.

Applying (1) under an assumption of small scale yielding we obtain

$$[u_p{}^{+}] \underset{x\to 0}{=} \frac{\tau_0}{\mu}(R-x)\sqrt{(1-\frac{x}{R})} \tag{7a}$$

which gives for the upper limit of the integral (6)

$$u_p{}^{+}(t) = \frac{\tau_0}{\mu}\left(R + \frac{\mathrm{d}R}{\mathrm{d}l}\Delta\right) \tag{8}$$

while for the lower limit we have,[9]

$$u_p{}^{+}(t-\delta t) = \frac{\tau_0}{\mu}(R-\alpha)\sqrt{\left(1-\frac{\Delta}{R}\right)}. \tag{8a}$$

Fracture criterion (6) now takes the form

$$R + \frac{\mathrm{d}R}{\mathrm{d}l}\Delta - (R-\Delta)\sqrt{\left(1-\frac{\Delta}{R}\right)} = \frac{u_0 \cdot \mu}{\tau_0} \tag{9}$$

while (7a) for $x = 0$ gives

$$u_p{}^{+}(0) = u_{\mathrm{tip}} = \frac{\tau_0}{\mu} R \text{ or } u_0 = (\tau_0/\mu)R_0 \tag{10}$$

in which R_0 is a new opening constant dependent on the final stretch u_0. Inserting (10) into (9) we obtain a non-linear differential equation which governs the quasi-static extension of a Mode III crack within the subcritical range of loading, namely

$$\frac{dR}{dl} = \frac{R_0 - R}{\Delta} + \left(\frac{R}{\Delta} - 1\right)\sqrt{\left(1 - \frac{\Delta}{R}\right)}. \tag{11}$$

In a dimensionless form we have

$$\frac{d\rho}{d\lambda} = \rho_0 - \rho + (\rho - 1)\sqrt{\left(1 - \frac{1}{\rho}\right)} \tag{12}$$

where

$$\rho = \frac{R}{\Delta}, \qquad \lambda = \frac{l}{\Delta}.$$

It is of interest to compare the initial slope of the R-curve given by eqn (12) with that predicted for Mode III by the incremental theory of plasticity, Rice[1]. From eqn (12) we have the initial slope

$$\left(\frac{d\rho}{d\lambda}\right)_0 = \sqrt{\left(\frac{\alpha}{\alpha + 1}\right)} \cdot \alpha \tag{13}$$

where $\alpha(=\rho_0 - 1)$ is the ductility parameter defined by McClintock[13] as the ratio of the plastic component of the shear strain at fracture to the strain at yield. Rice has

$$\left(\frac{d\rho}{d\lambda}\right)_0 = \alpha - \log(1 + \alpha).$$

Both results are plotted in Fig. 3. The third curve shown in Fig. 3 represents the initial slope of the resistance curve $(d\rho/d\lambda)_0$ predicted from the Wnuk's final stretch model. For the Mode III

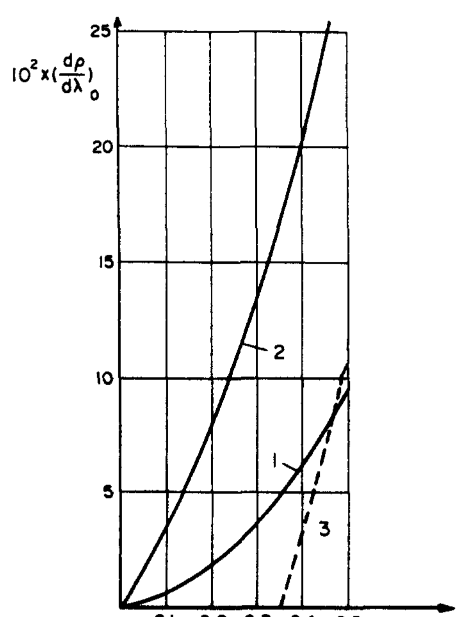


Fig. 3. Initial slope of the R-resistance curve for a Mode III crack as a function of the McClintock's ductility parameter α. 1. Rice's solution based on incremental plasticity, 2. Present solution, 3. Wnuk's solution based on the final stretch concept.

configuration Wnuk[12] obtained the following result

$$\frac{\mathrm{d}\rho}{\mathrm{d}\lambda} = \rho_0 - 1/2 \log (4e\rho) \tag{14a}$$

which implies the initial slope

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}\lambda}\right)_0 = \alpha - 1/2 - 1/2 \log [4(\alpha + 1)]. \tag{14b}$$

It might be of interest to point out a close resemblance of these "final stretch" solutions with those obtained by Wnuk[17, 18] and more recently by Rice and Sorensen[19] and Rice *et al.*[20] for the quasi-static tensile fracture. As it turns out, the only difference between the Mode I and Mode III solutions derived from the final stretch concept is in the definition of the constant ρ_0, i.e.

$$\rho_0 = \begin{cases} (\mu/\tau_0)(u_0/\Delta) & \text{for Mode III} \\ (\pi E_1/4Y)(u_0/\Delta) & \text{for Mode I} \end{cases} \tag{14c}$$

in which u_0 is the final stretch, Y denotes the effective yield stress encountered at the front of a propagating Mode I crack, while E_1 is the Young modulus E in plane stress and $E(1-\nu^2)^{-1}$ in plane strain. When these relations are taken into account, the eqn (14a) remains identical for both modes. Similarly, a remarkable analogy was found by Rice *et al.*[19, 20] in their final result for $\mathrm{d}\rho/\mathrm{d}\lambda$ and the form (14a) given in 1972 for the tensile fracture by one of the present authors.

3. CRACK GROWTH UNDER MONOTONIC LOADING

Let us find now the critical point at which the quasi-static motion of the crack, governed by eqn (12) changes over to a spontaneous and rapid propagation. For this purpose new variables are suggested, namely

$$\begin{aligned} x &= \log \lambda \\ y &= \log \rho, \quad y = y(x). \end{aligned} \tag{15}$$

We have

$$\frac{\mathrm{d}\rho}{\mathrm{d}\lambda} = \mathrm{e}^{y-x} \frac{\mathrm{d}y}{\mathrm{d}x} \tag{16}$$

while eqn (12) assumes the form†

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{x-y} \left[\mathrm{e}^{y_0} - \mathrm{e}^{y} + \sqrt{\left(\frac{\mathrm{e}^{y}-1}{\mathrm{e}^{y}}\right)(\mathrm{e}^{y}-1)}\right]. \tag{16a}$$

We have to keep in mind that the plastic zone size depends not only on the current crack length but also on the external load applied to material and represented here by the dimensionless loading parameter $Q(=\tau_\infty/\tau_0)$. Within the subcritical range of crack growth the loading parameter itself is a function of the crack length. Therefore, we have

$$\rho(\lambda) = \rho(\lambda, Q(\lambda)) \tag{17}$$

†In the numerical integration of eqns (12) and (16a) it has been assumed that the extent of the plastic zone at the onset of crack growth R_i equals the opening constant R_0. This is an approximation which ought to be verified experimentally. McClintock's[21] theory suggests a functional relation between both constants, namely

$$R_0 = (\Delta/2)[\log 4 + \sqrt{2}(R_i/\Delta) - 1].$$

Taking the first derivative of (17) we obtain

$$\frac{d\rho}{d\lambda} = \frac{\partial\rho}{\partial\lambda} + \frac{\partial\rho}{\partial Q}\cdot\frac{dQ}{d\lambda}. \tag{18}$$

At the critical point the crack continues to propagate at no further increase of load, thus

$$\left(\frac{dQ}{d\lambda}\right)_f = 0 \tag{19}$$

where the index "f" indicates "fracture". Combining eqns (18) and (19) gives

$$\left(\frac{d\rho}{d\lambda}\right)_f = \left(\frac{\partial\rho}{\partial\lambda}\right)_f. \tag{20}$$

For the geometrical configuration considered here eqn (1) yields

$$\frac{\partial\rho}{\partial\lambda} = 2Q^2 = \frac{\rho}{\lambda}. \tag{21}$$

This substituted back into eqn (16) implies that at the critical point the derivative dy/dx equals unity, namely

$$\left\{e^{x-y}\left[e^{y_0} - e^y + \sqrt{\left(\frac{e^y-1}{e^y}\right)}(e^y-1)\right]\right\}_f = 1 \tag{22}$$

The function dy/dx plotted vs the dimensionless crack length is shown in Fig. 4.

Now we compute the critical value of the loading parameter (Q_f) and the critical crack length (λ_f) which defines the function $Q(\lambda)$, i.e.

$$\frac{dQ}{d\lambda} = \frac{\rho_0 - 2Q^2(\lambda+1) + \dfrac{(2Q^2\lambda-1)^{3/2}}{Q\sqrt{2\lambda}}}{4Q\lambda}. \tag{23}$$

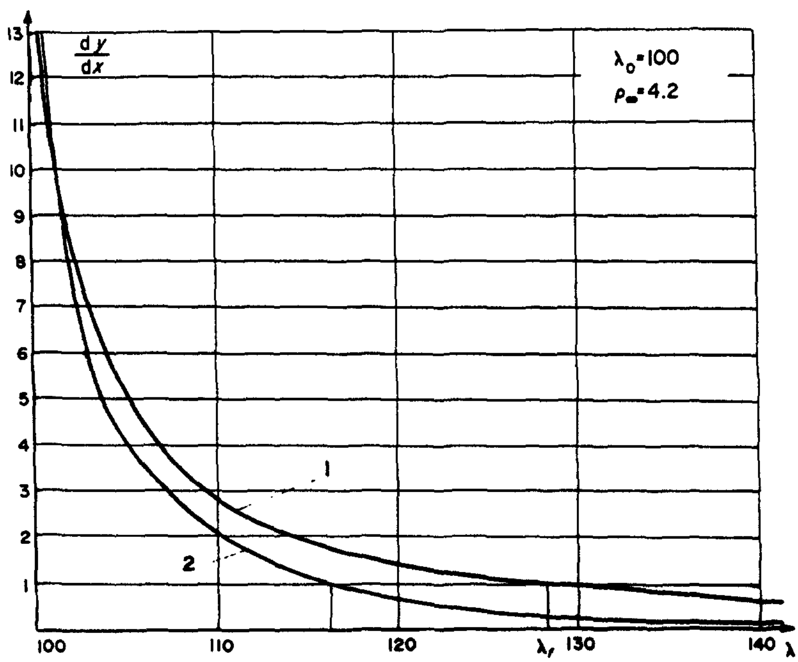


Fig. 4. (a) Slope of the resistance curve for Mode III fracture and (b) the R-resistance curve predicted by the present investigation (curve 1) and Wnuk's final stretch model (curve 2). Both graphs are drawn in non-dimensional coordinates ($\lambda = l/\Delta$, $\rho = R/\Delta$, $y = \log\rho$, $x = \log\lambda$).

At the point of transition to an unstable propagation the derivative $\mathrm{d}Q/\mathrm{d}\lambda$ vanishes resulting in the following locus of terminal instability

$$\rho_0 = \left\{2Q^2(\lambda+1) - \frac{(2Q^2\lambda - 1)^{3/2}}{Q\sqrt{(2\lambda)}}\right\}_f. \tag{24}$$

Computed critical values of Q and λ are listed in Table 1 while the integral curves $Q = Q(\lambda)$ obtained numerically from eqn (23) are shown in Fig. 5.

4. CRACK GROWTH UNDER CYCLIC LOADING

From Cherepanov's[7] and Wnuk's[6] theory of fatigue it follows that the the crack growth during cyclic loading could be viewed as a sequence of the subcritical and stable crack increments occurring within each successive load cycle. Since the amount of crack growth during a single cycle is a negligible fraction of the total crack length, it is reasonable to assume that the latter remains constant over a given loading cycle. This assumption facilitates somewhat the computational task aimed at the prediction of the fatigue crack growth rate, $\mathrm{d}\lambda/\mathrm{d}N$. To this end let us integrate the eqn (12) in a closed form. Omitting the algebraic details,

ρ_∞	λ_0	λ_f	Q_f	λ_0	λ_f	Q_f
3.0	100	120.5	0.110	50	64.4	0.136
4.2	100	129.6	0.115	50	69.3	0.145
10.0	100	153.7	0.133	50	80.6	0.163
20.0	100	169.0	0.141	50	86.9	0.170
100.0	100	186.0	0.148	50	93.3	0.176

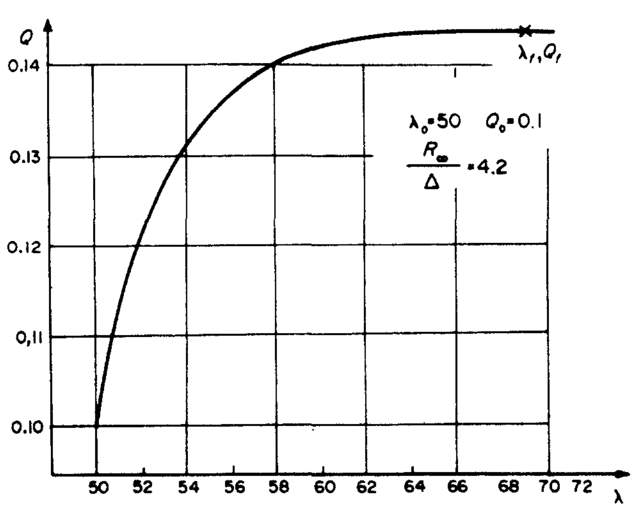


Fig. 5. Loading parameter $Q(=\tau_\infty/\tau_0)$ shown as a function of the current crack length $\lambda(=l/\Delta)$ during the subcritical phase of Mode III fracture.

we have

$$\lambda = \lambda_0 + (A_1 + A_2)\frac{v_0 - v}{(1-v)(1-v_0)} + \log\left[\left|\frac{1-v_0}{1-v}\right|^{A_2}\right.$$

$$\left.\cdot\left|\frac{1+v_0}{1+v}\right|^{A_3}\cdot\left|\frac{v_0 - v_1}{v - v_1}\right|^{A_4}\cdot\left|\frac{v_0 - v_2}{v - v_2}\right|^{A_5}\right], \quad v = \sqrt{\left(\frac{\rho - 1}{\rho}\right)}. \tag{25}$$

Finally, the fatigue crack extension rate is given by

$$\frac{\mathrm{d}\lambda}{\mathrm{d}N} = \left\{\frac{A_1 + A_2}{1 - v} + \log\left[|1 - v|^{A_2}\cdot|1 + v|^{A_3}\cdot|v - v_1|^{A_4}\cdot|v - v_2|^{A_5}\right]\right\}_{v_{\max}}^{v_{\min}} \tag{25a}$$

$$v_{\min} = \sqrt{\left(\frac{\rho_{\min} - 1}{\rho_{\min}}\right)} \qquad v_{\max} = \sqrt{\left(\frac{\rho_{\max} - 1}{\rho_{\max}}\right)}. \tag{25b}$$

The current length of crack does not alter appreciably during one cycle and therefore is considered constant in (25a). We have just derived the "logarithmic law" of fatique crack growth in anti-plane strain mode of deformation. Analogous to the equation of Cherepanov and Wnuk[7, 6]

$$\frac{\mathrm{d}l}{\mathrm{d}N} = -\beta\left[\frac{K_{\max}^2 - \delta K_{\min}^2}{K_c^2} + \log\frac{K_c^2 - K_{\max}^2}{K_c^2 - \delta K_{\min}^2}\right] \tag{26}$$

in which the empirical "crack closure factor" δ equals unity if the minimum load in the cycle falls above the crack closure threshold. In the numerical examples which we have appended here, the following relation has been used

$$\left(\frac{K_{\max}}{K_c}\right)^2 = \frac{\rho_{\max}}{\rho_\infty}. \tag{27}$$

We have also assumed that

$$\rho_{\min} = 1 \text{ (or } v_{\min} = 0).$$

Results are plotted in Figs. 7(a), (b) for two different materials distinguished by the constant ρ_∞.

From the mathematical limitations imposed on this model we have to require that both quantities, $\rho_{\min}$ and $\rho_{\max}$ are much greater than unity. For other values of ρ, i.e. for smaller $K_{\max}/K_c$ ratios and lesser ρ_∞ levels we have to extrapolate the function (25a). Material constants ρ_∞ and Δ should be estimated experimentally. In spite of lack of such experimental data we can

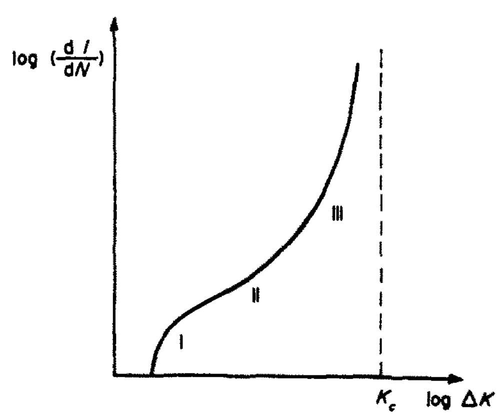


Fig. 6. Fatigue crack growth rate shown as a function of the stress intensity range.

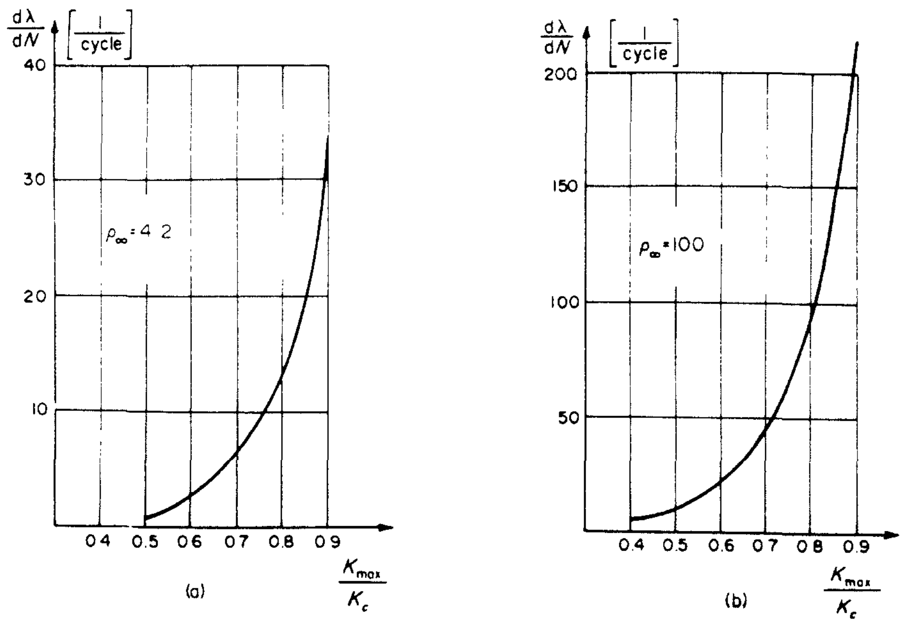


Fig. 7. Predicted $d\lambda/dN$ vs K_{max}/K_c curves for two materials; $\rho_\infty = 4.2$ and $\rho_\infty = 100$.

compare our results with those obtained by Cherepanov[8], Clark[10] and Forman[11]. The pertinent constants were chosen in such a way that the curves agree with analogous functions given by the previous authors.

5. CONCLUSIONS

The investigation outlined above makes it possible to draw the following conclusions:

1. The Dugdale model for anti-plane strain can be modified by a removal of the singularity in the stress component τ_{xz} in the vicinity of the crack tip.

2. In case of small scale yielding the stress intensity factor or the plastic zone size or Rice's J-integral can be used alternatively as the controlling factors for the quasi-static crack.

3. If the plastic zone length is chosen to represent the subcritical crack growth, then its characteristic values R_i (initial) and R_∞ (steady-state) are material properties.

4. The crack moves "jumping" with a constant velocity over Neuber's particle of length Δ, considered here to be a constant, too.

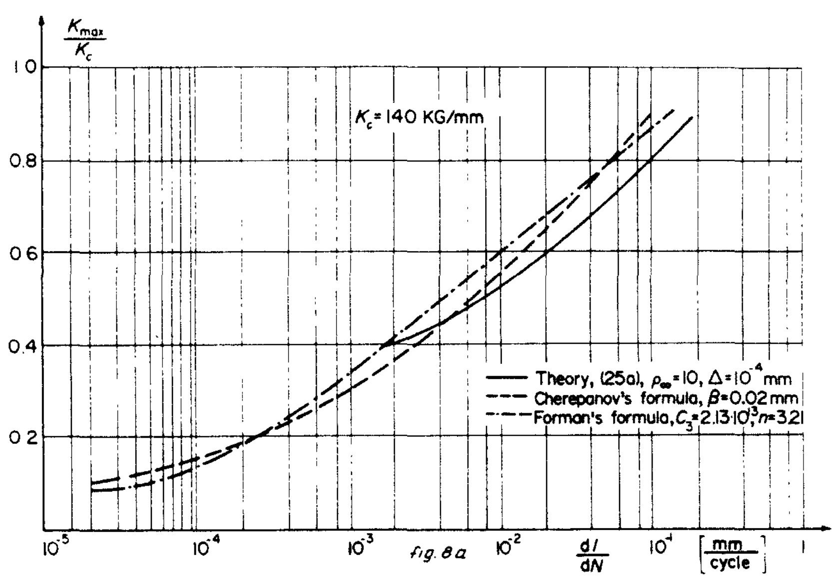
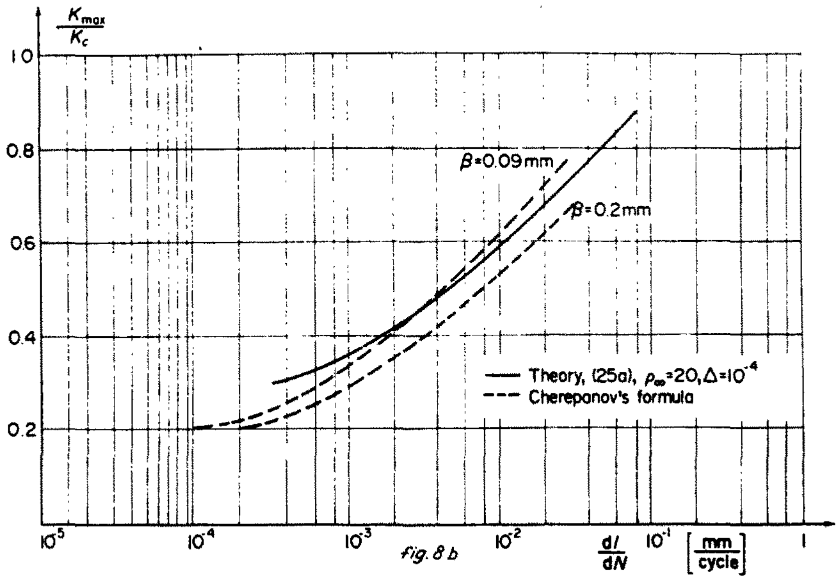
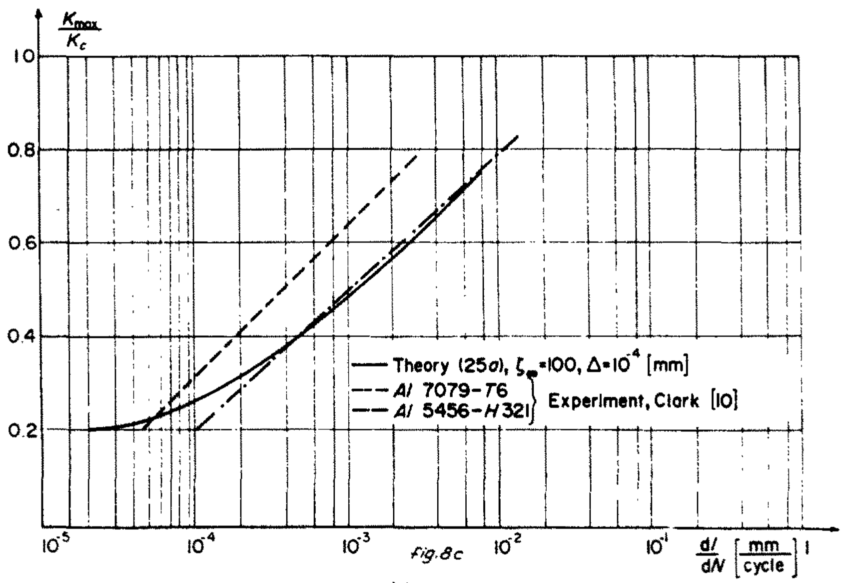


Fig. 8(a).



(b)



(c)

Fig. 8. Comparison of the experimental data on fatigue collected by Cherepanov[7], Clark[10], Forman[11] and the theoretical data derived from the present investigation.

5. Criterion of propagation is identical in monotonic and pulsating cases. It follows that the propagation exists even when the applied stress is much less than the stress necessary to cause rapid spread of fracture.

6. Transition to unstable crack propagation depends on (a) geometry, (b) ductility (ρ_∞) and "tearing modulus" (ρ_0) of the material, (c) initial and current crack length.

7. Within the small scale yielding range the R-resistance curve in Mode III is a unique material property independent of the geometrical and loading configurations just as it was found to be the case for the Mode I s.s.y. resistance curve.

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